Tutorial 5: Selected problems of Assignment 5

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§1: Jensen Inequality

Recall Jensen's Inequality:

Thm 1 (Theorem 1.10)

Let $f: I \to \mathbb{R}$ be a convex function, I: any interval

then for any $\lambda_1, \cdots, \lambda_n \in (0,1)$ such that $\sum_{j=1}^{n} \lambda_j = 1$,

for any $x_1, \cdots, x_n \in I$, we have

$$f(\lambda_1 x_1 + \cdots + \lambda_n x_n) \leq \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n)$$

("value of linear combination $\leq$ linear combination of values")

If f is strictly convex, then

Equality holds $\Leftrightarrow$ $x_1 = x_2 = \cdots = x_n$

Rmk Can replace "convex" by "concave", with reversed inequality instead.

Q1) (Supp. Ex. 3)

Show that for any $x, y, z \in (0, \pi)$ with $\frac{x+y+z}{3} = \frac{\pi}{3}$,

(a) $\sin x + \sin y + \sin z \leqslant \frac{3\sqrt{3}}{2}$

(b) $\sin x \sin y \sin z \leqslant \frac{3\sqrt{3}}{8}$

Show that $\underbrace{\text{each equality holds iff } x = y = z = \frac{\pi}{3}}_{(*)}$

Sol (a) Let $f : (0, \pi) \to \mathbb{R}$ be $f(\theta) = \sin\theta$

Then $f''(\theta) = -\sin\theta < 0$ $\therefore$ f is strictly concave

Apply Thm 1 with $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$:

$$\frac{1}{3}\sin x + \frac{1}{3}\sin y + \frac{1}{3}\sin z \geqslant \sin\left(\frac{x+y+z}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \sin x + \sin y + \sin z \leqslant \frac{3\sqrt{3}}{2}$$

Strict concavity $\Rightarrow$ $(*)$ is true.

(b) Method 1: by Jensen inequality

Taking log on both sides: suffices to show

$$\log\sin x + \log\sin y + \log\sin z \leq \log\left(\frac{3\sqrt{3}}{8}\right)$$

Let $f: (0,\pi) \to \mathbb{R}$ be $f(\theta) = \log\sin\theta$

Then $f''(\theta) = -\csc^2\theta < 0$. $\therefore$ f is strictly concave

Apply Thm 1 with $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$:

$$\frac{1}{3}(\log\sin x + \log\sin y + \log\sin z) \leq \log\sin\left(\frac{x+y+z}{3}\right) = \log\frac{\sqrt{3}}{2}$$

$$\therefore \log\sin x + \log\sin y + \log\sin z \leq 3\log\frac{\sqrt{3}}{2} = \log\frac{3\sqrt{3}}{8}$$

Strict concavity and log is strictly increasing $\Rightarrow$ (*) is true.

Method 2: AM-GM inequality:

$$(\sin x \sin y \sin z)^{\frac{1}{3}} \leq \frac{\sin x + \sin y + \sin z}{3} \leq \frac{\sqrt{3}}{2}$$

$$\therefore \sin x \sin y \sin z \leq \frac{3\sqrt{3}}{8}$$

Equality condition of AM-GM $\Rightarrow$ (*) is true

Q2) (Supp. Ex. 4)

Show that for $a, b, c > 0$,

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}$$

Sol Taking log on both sides, suffices to show

$$a \log a + b \log b + c \log c \geq (a+b+c) \log\left(\frac{a+b+c}{3}\right)$$

Let $f : (0, \infty) \to \mathbb{R}$ be $f(x) = x \log x$

Then $f''(x) = \frac{1}{x} > 0$ $\therefore$ f is strictly convex

Apply Thm 1 with $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$:

$$\frac{1}{3}(a \log a + b \log b + c \log c) \geq \frac{(a+b+c)}{3} \log\left(\frac{a+b+c}{3}\right)$$

§2: Integrability Criterions

Let $f : [a,b] \to \mathbb{R}$ be bounded,

Recall Integrability Criterion I :

Thm 2 (Theorem 2.5)

$$f \in R[a,b] \iff \overline{S}(f) = \underline{S}(f)$$

where $R[a,b] = \{\text{Riemian-Integrable functions on } [a,b]\}$

$$\overline{S}(f) = \inf_P \overline{S}(f;P)$$

$$\underline{S}(f) = \sup_P \underline{S}(f;P)$$

If so, define $\int_a^b f := \overline{S}(f) = \underline{S}(f)$

Also, recall Integrability Criterion II

Thm 3 (Theorem 2.7)

$$f \in R[a,b] \iff \forall \varepsilon > 0, \exists \text{ partition } R \text{ such that}$$

$$\overline{S}(f;R) - \underline{S}(f;R) < \varepsilon$$

Q3 (§7.1 Q 11)

If $\exists$ $\{\dot{P}_n\}$, $\{\dot{Q}_n\}$, such that $\lim_n \|\dot{P}_n\| = 0 = \lim_n \|\dot{Q}_n\|$

and $\lim_n S(f;\dot{P}_n) \neq \lim_n S(f;\dot{Q}_n)$

then $f \notin R[a,b]$

Pf WLOG assume $\lim_n S(f;\dot{P}_n) > \lim_n S(f;\dot{Q}_n)$,

$$\overline{S}(f) = \lim_n \overline{S}(f;P_n) \quad (\text{Thm } 2.4')$$

$$\geq \lim_n S(f;\dot{P}_n) \quad (\because \overline{S}(f;P_n) \geq S(f;\dot{P}_n), \forall n)$$

$$> \lim_n S(f;\dot{Q}_n) \quad (\text{Assumption})$$

$$\geq \lim_n \underline{S}(f;Q_n) \quad (\because \underline{S}(f;Q_n) \leq S(f;\dot{Q}_n), \forall n)$$

$$= \underline{S}(f) \quad (\text{Thm } 2.4')$$

$\therefore$ $\overline{S}(f) > \underline{S}(f)$. By Thm 2, $f \notin R[a,b]$.

Q4) (Supp. Ex. 6)

Prove Cauchy Criterion for integrability :

$f \in R[a,b] \iff \forall \varepsilon > 0, \exists \delta > 0$ such that

for all $\dot{P}, \dot{Q}$ with $|\dot{P}|, |\dot{Q}| < \delta$,

$|S(f;\dot{P}) - S(f;\dot{Q})| < \varepsilon$

Sol [$\Rightarrow$] Let $\varepsilon > 0$ be given.

$f \in R[a,b] \Rightarrow \exists L \in \mathbb{R}$ such that $\exists \delta > 0$,

$\forall \dot{R}$ with $|\dot{R}| < \delta$, $|S(f;\dot{R}) - L| < \frac{\varepsilon}{2}$

Therefore, $\forall \dot{P}, \dot{Q}$ with $|\dot{P}|, |\dot{Q}| < \delta$,

$$|S(f;\dot{P}) - S(f;\dot{Q})| \leq |S(f;\dot{P}) - L| + |L - S(f;\dot{Q})|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

[⇐] By Thm 3 : Given $\varepsilon > 0$, suffices to find R s.t.

$$\overline{S}(f;R) - \underline{S}(f;R) < \varepsilon$$

By assumption, $\exists\, \delta > 0$ s.t. $\forall\, \dot{P}, \dot{Q}$ with $|\dot{P}|, |\dot{Q}| < \delta$,

$$|S(f;\dot{P}) - S(f;\dot{Q})| < \frac{\varepsilon}{2} \quad -(**)$$

Pick any R with $|R| < \delta$. By Prop. 2.3,

there exists two tags $\dot{R}, \ddot{R}$ such that

$$\begin{cases} \overline{S}(f;R) - S(f;\dot{R}) \leq \frac{\varepsilon}{4} \\ S(f;\ddot{R}) - \underline{S}(f;R) \leq \frac{\varepsilon}{4} \end{cases}$$

Also, $|S(f,\dot{R}) - S(f;\ddot{R})| < \frac{\varepsilon}{2}$ (by $(**)$)

Therefore, $\overline{S}(f,R) - \underline{S}(f,R)$

$$= (\overline{S}(f;R) - S(f;\dot{R})) + (S(f;\dot{R}) - S(f;\ddot{R})) + (S(f;\ddot{R}) - \underline{S}(f;R))$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon$$